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Singular left-definite Sturm–Liouville problems

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Abstract

We study singular left-definite Sturm–Liouville problems with an indefinite weight function. The existence of eigenvalues is established based on the existence of eigenvalues of corresponding right-definite problems. Furthermore, for each singular left-definite problem with limit-circle non-oscillatory endpoints we construct a regular left-definite problem with the same eigenvalues and use it to obtain properties of eigenvalues and eigenfunctions. Inequalities among eigenvalues recently established for regular left-definite problems are extended to the singular case.

MSC: primary 34B24, 34L15; secondary 34L05

Keywords: Singular problems; Left-definite; Eigenvalues; Eigenfunctions; Eigenvalue inequalities

1. Introduction

We study singular left-definite (LD) Sturm–Liouville problems (SLP's) for the equation

$$My := -(py')' + qy = \lambda w y \text{ on } J = (a, b), \ -\infty \leqslant a < b \leqslant \infty, \tag{1.1}$$

where the weight function w changes sign. The existence of eigenvalues is established by elementary means, i.e. without using “LD Hilbert space” or Krein space operator theory.

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Our approach is based on the method of "spectral-curves" generated by the equation

$$-(py')' + (q - \lambda w)\, y \; = \xi\, |\, w|\; y \;\text{ on } J \tag{1.2}$$

and a self-adjoint domain, where ξ and λ are spectral parameters. For each $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$, $\xi_n(\lambda)$ is defined by applying the max-min principle from the right-definite (RD) theory to the one parameter family of Eq. (1.2), and $\xi = \xi_n(\lambda), \lambda \in \mathbb{R}$, is called the *n*th spectral-curve of the problem. These spectral-curves, obtained from the right-definite theory, yield information about eigenvalues for LD problems associated with Eq. (1.1).

For regular problems these spectral-curves reduce to eigencurves. Such eigencurves for regular problems on a bounded interval J with separated boundary conditions (BC's), and with $|w|$ replaced by 1, have been investigated by other authors, see the recent papers Binding and Volkmer [6], Binding and Browne [3–5]. While we have been strongly influenced by the work of these authors, our work differs from theirs in several respects: the BC's can be separated or coupled, the problems can be singular with limit-point and/or limit-circle endpoints, and the weight function in Eq. (1.2) is $|w|$ rather than 1. The "interplay" between w and $|w|$ in Eq. (1.2) has significant advantages, particularly for singular problems. See Lemma 4.2 and Remark 4.1.

The special case $\lambda = 0$ in Eq. (1.2) plays a critical role, so we highlight the equation

$$My = -(py')' + qy = \xi\, |\, w|\, y \;\text{ on }\; J. \tag{1.3}$$

We call Eq. (1.3) the RD equation associated with Eqs. (1.1) and (1.2) the one parameter family of RD equations associated with Eq. (1.1).

This paper can be considered a follow-up of the recent paper [17] by Kong et al. in which the regular LD problems associated with Eq. (1.2) were investigated. In this paper, we establish the existence of eigenvalues of singular LD problems in terms of the existence of eigenvalues of corresponding singular RD problems below their essential spectra using the spectral-curve approach. Furthermore, for each singular LD problem with limit-circle non-oscillatory endpoints we construct a *regular* LD problem with the same eigenvalues. This is accomplished by employing a transformation similar to the one used by Niessen and Zettl [22] in the RD case. This allows us to extend many of the results from the regular to the singular case, in particular, those recently established in [17].

A special class of singular LD problems with limit-circle non-oscillatory endpoints and a specific singular BC was studied by Kaper et al. [14]. For a "LD Hilbert space" approach to the study of singular boundary value problems with an indefinite weight function see Vonhoff [25,26], Bennewitz and Everitt [2], and the references cited therein; for a Krein space approach to LD and indefinite problems see Curgus and Langer [8,9] and the reference cited there. Also see the seminal paper of Weyl [28].

The organization of this paper is as follows: Following this introduction, notation and basic results are given in Section 2. Section 3 contains the main existence theorem; its proof along with some technical lemmas are given in Section 4. In Section 5 we specialize to the case of non-oscillatory limit-circle endpoints; further results for this case are given in Section 6 on the eigenvalue inequalities, asymptotic behavior and ranges of eigenvalues.

2. Notation and basic results

Let

$$\mathbb{N}_0 = \{0, 1, 2, ...\} \quad \text{and} \quad \mathbb{Z}^* = \{-2, -1, -0, 0, 1, 2, ...\}.$$

For any subinterval I of $\mathbb{R}$, we denote by $L(I;\mathbb{C})$ the set of complex-valued Lebesgue integrable functions on I, and by $L_{\text{loc}}(I;\mathbb{C})$ the set of complex-valued functions on I which are Lebesgue integrable on each compact subinterval of I. Similar definitions are made for $L(I;\mathbb{R})$ and $L_{\text{loc}}(I;\mathbb{R})$.

In a seminal paper [24] Sims extended the well-known limit-point (LP), limit-circle (LC) dichotomy of Weyl for real coefficients to a trichotomy for a complex potential function q. This was further extended for complex p, q and real and positive w by Brown et al. [7]. Although we study only the real-valued coefficient case in this paper, we use an LC/LP dichotomy in the spirit of Weyl for the general two-parameter equation (1.2) with complex coefficients.

Definition 2.1. Let

$$1/p,\ q,\ w \in L_{\text{loc}}(J;\mathbb{C}), \qquad w \neq 0 \text{ a.e. on } J. \tag{2.1}$$

The endpoint a for Eq. (1.2) is in the *LC case*, or simply a is LC, if for all $\lambda, \xi \in \mathbb{C}$ and for some $c \in J$, all solutions of Eq. (1.2) are in $L^2((a, c), |w|)$; otherwise, a is in the *LP case*, or simply a is LP. Similar definitions are made for the endpoint b.

If for all $\{\lambda, \xi\} \in \mathbb{C}$, both a and b are LC (resp. LP), then we say that Eq. (1.2) is LC (resp. LP).

Remark 2.1. (1) In view of (2.1) the LC/LP definition is clearly independent of $c \in J$. The next lemma will show that a is LC provided for some $\lambda, \xi \in \mathbb{C}$ and $c \in J$, all solutions of (1.2) are in $L^2((a, c), |w|)$. Similarly for b. Hence the LC/LP classification of a given endpoint depends only on $1/p, q, w$. Note that no conditions other than (2.1) are imposed on these coefficients in Lemma 2.1 below. So each of $1/p, q, w$ can be complex-valued and, if real-valued, it may change sign. Also, each of $1/p, q, w$ can be identically zero on one or more subintervals of J.

(2) If a is a regular endpoint, i.e.,

$$1/p,\ q,\ w \in L((a,c);\mathbb{C}) \quad \text{for some } c \in J,$$

then a is LC. The same for b. In general, a LC endpoint is not a regular endpoint.

Lemma 2.1. *Let (2.1) hold. Let $c \in J$. Assume that all solutions of Eq. (1.2) are in $L^2((a,c),|w|)$ for some pair $\{\lambda,\xi\} = \{\lambda^*,\xi^*\} \in \mathbb{C}^2$. Then this is true for all pairs $\{\lambda,\xi\} \in \mathbb{C}^2$. Similarly for the endpoint b.*

Proof. From the RD theory it is well-known that if all solutions of Eq. (1.2) are in $L^2((a,c),|w|)$, for some pair $\{\lambda,\xi\} = \{\lambda^*,\xi^*\} \in \mathbb{C}^2$, then this is true for all pairs $\{\lambda,\xi\} = \{\lambda^*,\xi\} \in \mathbb{C}^2$. Therefore, it is sufficient to show that if all solutions are in $L^2((a,c),|w|)$ for some $\{\lambda^*,\xi^*\} \in \mathbb{C}^2$, then this is true for all $\{\lambda,\xi^*\}$ with $\lambda \in \mathbb{C}$.

To prove this, let u_1, u_2 be linearly independent solutions of Eq. (1.2) with $\{\lambda,\xi\} = \{\lambda^*,\xi^*\}$ such that $\det\begin{pmatrix} u_1 & u_2 \\ pu_1' & pu_2' \end{pmatrix}(t) \equiv 1$. Then $u_1, u_2 \in L^2((a,c),|w|)$ for $c \in J$. Let

$$Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1/p \\ q - \xi^*|w| & 0 \end{pmatrix}, \quad \text{and } W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}.$$

Then the system formulation of equation (1.2) with $\xi = \xi^*$ is given by

$$Y' = (P - \lambda W)Y. \tag{2.2}$$

Note that $U = \begin{pmatrix} u_1 & u_2 \\ pu_1' & pu_2' \end{pmatrix}$ is a fundamental matrix solution of Eq. (2.2) for $\lambda = \lambda^*$ with $\det U(t) \equiv 1$. For any vector or matrix solution $Y(t,\lambda)$ of Eq. (2.2) we define

$$X(t,\lambda) = U^{-1}(t)Y(t,\lambda). \tag{2.3}$$

Then a routine computation shows that $X(t,\lambda)$ is a vector or matrix solution of the equation

$$X' = (\lambda^* - \lambda)GX \quad \text{on } J, \tag{2.4}$$

where

$$G = U^{-1}WU = \begin{pmatrix} -u_1u_2w & -u_2^2w \\ u_1^2w & u_1u_2w \end{pmatrix}.$$

It follows from the hypotheses and the Schwarz inequality that $G \in L((a,c);\mathbb{C}^{2\times 2})$. Hence Eq. (2.4) is regular at a. This implies that every solution $X(t)$ has a finite limit at a and hence is bounded on (a,c). Note from (2.3) that Y is a solution of Eq. (2.2) if and only if X is a solution of Eq. (2.4). Hence for every scalar solution y of Eq. (1.2) there exists a vector solution $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ of Eq. (2.4) such that

$$y = u_1 x_1 + u_2 x_2.$$

Hence $y \in L^2((a,c),|w|)$.

The proof for the endpoint b is similar. □

Corollary 2.1. *If all solutions of Eq.* (1.2) *are in* $L^2(J,|w|)$ *for some pair* $\{\lambda,\xi\} = \{\lambda^*,\xi^*\} \in \mathbb{C}^2$*, then this is true for all pairs* $\{\lambda,\xi\} \in \mathbb{C}^2$.

Proof. This is an immediate consequence of Lemma 2.1. □

In the rest of this paper we assume that the coefficients of Eq. (1.1) satisfy the basic conditions

$$1/p,\ q,\ w \in L_{\text{loc}}(J;\mathbb{R}), \quad p,\ |w| > 0 \text{ a.e. on } J, \quad w \text{ changes sign on } J. \tag{2.5}$$

Here and below "w changes sign on J" means that w assumes positive and negative values on subsets of J with positive or infinite Lebesgue measure. Such a w is also referred to as "indefinite".

Throughout this paper we let $H = L^2(J,|w|)$ be the Hilbert space with the inner product $(f,g) = \int_a^b f\,\overline{g}\,|w|$, $AC_{\text{loc}}(J,\mathbb{C})$ the set of complex valued functions which are absolutely continuous on each compact subinterval of J.

Consider the maximal and minimal domains and maximal and minimal operators associated with Eq. (1.2) given by

$$D_{\max}(\lambda) = \left\{ f \in H:\ f,\ pf' \in AC_{\text{loc}}(J;\mathbb{C}),\ (Mf - \lambda w f)/|w| \in H \right\},$$
$$S_{\max}(\lambda) f = (Mf - \lambda w f)/|w| \quad \text{for } f \in D_{\max}(\lambda)$$

and

$$S_{\min}(\lambda) = S^*_{\max}(\lambda), \quad D_{\min}(\lambda) = \text{the domain of } S_{\min}(\lambda), \tag{2.6}$$

where $S^*_{\max}(\lambda)$ denotes the adjoint of the operator $S_{\max}(\lambda)$ in H.

By a self-adjoint realization of Eq. (1.2) in H we mean an operator $S(\lambda)$ satisfying

$$S_{\min}(\lambda) \subset S(\lambda) = S^*(\lambda) \subset S_{\max}(\lambda).$$

Such a self-adjoint realization is determined completely by its domain and can be viewed as either a restriction of the maximal operator or an extension of the minimal operator. We will refer to the domains of self-adjoint realizations as self-adjoint domains.

The special case $\lambda = 0$ plays an important role; for this case we use the notation $S_{\max} := S_{\max}(0)$, $S_{\min} := S_{\min}(0)$, and S for arbitrary self-adjoint realizations $S(0)$.

Lemma 2.2. *For each $\lambda \in \mathbb{R}$ we have*

$$D_{\max}(\lambda) = D_{\max}(0) \text{ and } D_{\min}(\lambda) = D_{\min}(0).$$

Hence they can be written as $D_{\max}$ and $D_{\min}$, respectively.

If S is a self-adjoint realization of Eq. (1.3) *in H with domain D, then for each $\lambda \in \mathbb{R}$ the operator $S(\lambda) : D \to H$ defined by*

$$S(\lambda)f = (Mf - \lambda wf)/|w|, \quad f \in D, \tag{2.7}$$

is a self-adjoint realization of Eq. (1.2), *and all self-adjoint realizations of Eq.* (1.2) *are obtained this way, i.e., they are determined by the domain of a self-adjoint realization S of Eq.* (1.3).

Furthermore, if $S(\lambda)$ is bounded below for some $\lambda \in \mathbb{R}$, then it is bounded below for every $\lambda \in \mathbb{R}$.

Proof. From its definition we see that the maximal domain $D_{\max}(\lambda)$ is independent of λ. Hence the minimal operator $D_{\min}(\lambda)$ is also independent of λ.

Suppose S is self-adjoint in H with domain D. Define an operator $\Phi(\lambda) : H \to H$ by $\Phi(\lambda)f = \lambda(\operatorname{sgn} w)f$. Then $\Phi(\lambda)$ is self-adjoint on H for any $\lambda \in \mathbb{R}$. We observe that $S(\lambda) = S - \Phi(\lambda)$. Hence $S(\lambda)$ is self-adjoint on D. Conversely, if $S(\lambda)$ is self-adjoint in H with domain D, then S is self-adjoint with domain D.

To prove the furthermore statement assume that for some $\lambda = \lambda^*$, there exists $c \in \mathbb{R}$ such that $(S(\lambda^*)f, f) \geqslant c(f, f)$ for all $f \in D$. Then for any $\lambda \in \mathbb{R}$ and $f \in D$

$$(S(\lambda)f, f) = \left(\frac{1}{|w|}(Mf - \lambda^* wf), f\right) - \left((\lambda - \lambda^*)\frac{w}{|w|}f, f\right) \geqslant (c - |\lambda - \lambda^*|)(f, f),$$

i.e., $S(\lambda)$ is bounded below by $c - |\lambda - \lambda^*|$. □

Remark 2.2. It follows from Lemma 2.2 that the self-adjoint domains are invariant with respect to $\lambda \in \mathbb{R}$ and thus, in particular, are given by those when $\lambda = 0$. Note however that, although their domains are independent of λ, the self-adjoint operators $S(\lambda)$ depend on λ.

For the convenience of the reader we state the well-known min–max principle for RD problems, see [10, p. 1543], which plays an important role in this paper. We denote by $G_n(V)$ the set of n-dimensional subspaces of a vector space V.

Lemma 2.3. *Suppose $S : D \to H$ is a self-adjoint operator and is bounded below. Let $\sigma_e(S)$ be the essential spectrum of S. For each $n \in \mathbb{N}_0$, define*

$$s_n(S) = \inf \left\{ \sup\{(Sf, f) : f \in F \cap U\} : F \in G_{n+1}(D) \right\},$$

where U is the unit sphere in H. Then for any fixed $n \in \mathbb{N}_0$, either

$s_n(S) = \inf \sigma_e(S)$, *in this case,* $s_n(S) = s_{n+1}(S) = s_{n+2}(S) = \cdots$, *there are at most $n+2$ eigenvalues of S, counting multiplicity, less than or equal to* $\inf \sigma_e(S)$, *and these eigenvalues are among $s_0(S)$, $s_1(S)$, ..., $s_{n+1}(S)$; or*

$s_n(S) < \inf \sigma_e(S)$, *in this case, there are at least $n+1$ eigenvalues $s_0(S)$, $s_1(S)$, ..., $s_n(S)$* of *S, counting multiplicity, strictly less than* $\inf \sigma_e(S)$ (*or ∞ if $\sigma_e(S) = \emptyset$*), *and $s_n(S)$ is the $(n+1)$th eigenvalue of S, counting multiplicity.*

Although the characterization of the self-adjoint domains in terms of BC's is well-known, see [29,16] for details, we summarize it here for the convenience of the reader and also to make more explicit the singular BC's for Eq. (1.1) to which our results apply. The number of BC's needed and allowed to determine self-adjoint domains depends on the LC/LP classification of the endpoints. Recall that the Lagrange sesquilinear form is given by

$$[f, g] = f(p\overline{g}') - \overline{g}(pf'), \quad f, \ g \in D_{\max}.$$

Proposition 2.1. *The* BC's *determining self-adjoint domains for Eq.* (1.3) *are as follows*:

1. *Suppose both endpoints are* LP. *In this case, no* BC's *are needed or allowed and $S_{\min}$ is self-adjoint and has no proper self-adjoint extensions in H.*

2. *Suppose a is* LC *and b is* LP. *In this case, there exist real-valued $u, v \in D_{\max}$ such that $[u, v](a) \neq 0$. For $\alpha \in [0, \pi)$, define*

$$D = \{y \in D_{\max} : \cos\alpha \, [y, u](a) - \sin\alpha \, [y, v](a) = 0\}.$$

Then D is a self-adjoint domain and all self-adjoint domains are obtained this way.

3. *Suppose a is* LP *and b is* LC. *In this case, there exist real-valued $u, v \in D_{\max}$ such that $[u, v](b) \neq 0$. For $\beta \in (0, \pi]$, define*

$$D = \{y \in D_{\max} : \cos\beta \, [y, u](b) - \sin\beta \, [y, v](b) = 0\}.$$

Then D is a self-adjoint domain and all self-adjoint domains are obtained this way.

4. *Suppose both endpoints are* LC. *In this case, there exist real-valued* $u, v \in D_{\max}$ *such that* $[u, v](a) = [u, v](b) = -1$. *Define*

$$D = \{y \in D_{\max} : AY(a) + BY(b) = 0\}$$

where $Y = \begin{pmatrix} [y, u] \\ [y, v] \end{pmatrix}$ *and* A, B *are* 2×2 *complex matrices satisfying*

$$AEA^* = BEB^* \ \textit{with} \ \operatorname{rank}(A|B) = 2 \ \textit{and} \ E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.8}$$

here A^*, B^* *are the complex conjugate transposes of* A, B, *respectively. Then* D *is a self-adjoint domain and all self-adjoint domains are obtained this way*.

The BC's *given in Part* 4 *can be classified into two mutually exclusive classes: separated and coupled. The former has the canonical representation*:

$$\begin{aligned} &\cos\alpha\,[y, u](a) - \sin\alpha\,[y, v](a) = 0, \quad \alpha \in [0, \pi), \\ &\cos\beta\,[y, u](b) - \sin\beta\ [y, v](b) = 0, \ \ \beta \in (0, \pi] \end{aligned} \tag{2.9}$$

and the latter has the canonical representation:

$$Y(b) = e^{i\gamma} K\, Y(a), \tag{2.10}$$

where $K \in \mathrm{SL}(2, \mathbb{R}) := \{K \in \mathbb{R}^{2\times 2} : \det K = 1\}$ *and* $\gamma \in [0, \pi)$. *The coupled condition* (2.10) *is said to be real if* $\gamma = 0$ *and is called non-real otherwise*.

Remark 2.3. (i) The "BC functions" $\{u, v\}$ in Proposition 2.1 can be viewed as a basis in BC space and the "BC constants" α, β or K, γ, as the coordinates of a specific BC with respect to a given basis $\{u, v\}$. Given a fixed basis $\{u, v\}$, by varying the BC coordinates through their prescribed ranges, all self-adjoint domains are obtained. Thus, the same self-adjoint domain D has different characterizations depending on which BC basis $\{u, v\}$ is used. How do the BC coordinates change when the BC basis $\{u, v\}$ changes? This question is easy to answer when one endpoint is LP, and has been answered by Theorem 3.3 in [16] when both endpoints are LC.

(ii) If a is a regular endpoint, then the BC basis $\{u, v\}$ can be chosen to satisfy the initial conditions : $u(a) = 0,\ (pu')(a) = 1;\ v(a) = 1,\ (pv')(a) = 0;$ in this case, the BC in case 2 of Proposition 2.1 reduces to the more familiar form

$$\cos\alpha\, y(a) - \sin\alpha\,(py')(a) = 0, \ \ \alpha \in [0, \pi).$$

Similar for the case that b is a regular endpoint.

If both endpoints are regular, using the Naimark patching lemma we may construct a BC basis $\{u, v\}$ satisfying that for $t = a, b$

$$u(t) = 0,\ (pu')(t) = 1;\ v(t) = 1,\ (pv')(t) = 0.$$

In this case, (2.9) and (2.10) take the more familiar forms

$$\begin{aligned} \cos\alpha\, y(a) - \sin\alpha\,(py')(a) &= 0, \quad \alpha \in [0, \pi), \\ \cos\beta\, y(b) - \sin\beta\ (py')(b) &= 0, \quad \beta \in (0, \pi] \end{aligned}$$

and

$$Y(b) = e^{i\gamma} K\, Y(a),\ Y = \begin{pmatrix} y \\ py' \end{pmatrix},$$

respectively.

3. Existence of eigenvalues

In this section we establish the existence of real eigenvalues for singular LD SLP's associated with Eq. (1.1).

Let S be a self-adjoint realization of Eq. (1.3) in H with domain D. We define functionals $\mathcal{L}$ and $\mathcal{R}$ from D to $\mathbb{R}$ by

$$\mathcal{L}f = \int_a^b (Mf)\bar{f},\ f \in D \quad \text{and} \quad \mathcal{R}f = \int_a^b |f|^2 w,\ f \in D. \tag{3.1}$$

We denote by U the unit sphere in the Hilbert space H.

We now state the definition of LD SLP's.

Definition 3.1. Let S be a self-adjoint realization of Eq. (1.3) in H with domain D. Denote by $\{(1.1), D\}$ the SLP consisting of Eq. (1.1) and the BC which characterize D. Then the SLP $\{(1.1), D\}$ is *left-definite* (LD) if there exists a number $c > 0$ such that

$$\mathcal{L}f \geqslant c \quad \text{for all } f \text{ in } D \cap U. \tag{3.2}$$

Remark 3.1. There are various definitions of left-definiteness in the literature. Littlejohn and Wellman [19] give a general abstract definition of LD operators which are bounded below; this rules out SLP's with indefinite weight functions. Vonhoff [25,26]—following Pleijel [23], Bennewitz and Everitt [2], Everitt [12], Niessen and Schneider [20,21] and others, define left-definiteness for indefinite weight functions by constructing a Sobolev type Hilbert space based on the left-hand side of Eq. (1.1) and then using the operator theory in this space rather than in the L^2 space. In [25] the self-adjoint realizations in the Sobolev type Hilbert space are characterized, in general, by λ-dependent BC's. Many of the above definitions apply also to the problems with positive weight functions. In this paper, under the assumption (2.5) we define the left-definiteness only for problems with indefinite weight functions. This is done solely for convenience in stating the results. Clearly, our definition could also be used for the case when $w > 0$ a.e. on J.

Lemma 3.1. *Let $\lambda \in \mathbb{R}$. Then λ is in the spectrum of $\{(1.1), D\}$, i.e., the spectrum of $\frac{1}{w}M : D \to H$, if and only if $0 \in \sigma(S(\lambda))$.*

Proof. Note that the linear operator S is self-adjoint and hence closed. So, $S(\lambda)$ and

$$\frac{1}{w}M = (\operatorname{sgn} w)\frac{1}{|w|}M = (\operatorname{sgn} w)S : D \to H$$

are also closed. Thus, by (2.7),

$$\begin{aligned} 0 \notin \sigma(S(\lambda)) &\iff S(\lambda) = \frac{1}{|w|}M - \lambda \operatorname{sgn} w : D \to H \text{ is 1-1 and onto} \\ &\iff \frac{1}{w}M - \lambda = (\operatorname{sgn} w)S(\lambda) : D \to H \text{ is 1-1 and onto} \\ &\iff \lambda \text{ is not in the spectrum of } \frac{1}{w}M : D \to H. \qquad \square \end{aligned}$$

Lemma 3.2. *Assume the* SLP *$\{(1.1), D\}$ is LD. Then, its spectrum is real, and 0 is not in the spectrum.*

Proof. For $f \in D$, $(Sf, f) = \mathcal{L}f$. Thus, (3.2) implies that c is a lower bound for the self-adjoint operator S, and hence is also a lower bound for the spectrum $\sigma(S)$. Since $c > 0$, $0 \notin \sigma(S)$. By Lemma 3.1 with $\lambda = 0$, 0 is not in the spectrum of $\{(1.1), D\}$. Hence, Corollary 4.3 in [15] implies that the spectrum of $\{(1.1), D\}$ is real. $\square$

Definition 3.2. Let $S(\lambda)$, $\lambda \in \mathbb{R}$, be a self-adjoint realization of Eq. (1.2) given by (2.7) with domain D. Then for each $n \in \mathbb{N}_0$

$$\xi_n(\lambda) = \inf\left\{\sup\{(S(\lambda)f, f) : f \in F \cap U\} : F \in G_{n+1}(D)\right\} \tag{3.3}$$

is defined for all $\lambda \in \mathbb{R}$. We call $\xi = \xi_n(\lambda)$, $\lambda \in \mathbb{R}$, the nth *spectral-curve* of S.

If for each $\lambda \in \mathbb{R}$, $\xi_n(\lambda)$ is an eigenvalue of $S(\lambda)$, then $\xi = \xi_n(\lambda), \lambda \in \mathbb{R}$, is called an *eigencurve* of $S(\lambda)$.

By Lemma 2.2, if S is bounded below, then $\xi_n(\lambda)$ is finite for each $n \in \mathbb{N}_0$ and every $\lambda \in \mathbb{R}$. In general, a spectral-curve may contain some eigenvalues, and at the same time, some essential spectral points. Hence, it may or may not be an eigencurve.

We observe that for the operator $S(\lambda)$ defined by (2.7) and for any $f \in U$

$$(S(\lambda)f, f) = \mathcal{L}f - \lambda\mathcal{R}f.$$

Hence (3.3) becomes

$$\xi_n(\lambda) = \inf\left\{\sup\{\mathcal{L}f - \lambda\mathcal{R}f : f \in F \cap U\} : F \in G_{n+1}(D)\right\}. \tag{3.4}$$

Note that λ *is an eigenvalue of* $\{(1.1), D\}$ *if and only if* $\xi_n(\lambda) = 0$ *for some* $n \in \mathbb{N}_0$, *and 0 is an eigenvalue of the operator* $S(\lambda)$*; in this case, these two eigenvalues have the same eigenspace*.

Lemma 3.3. *The SLP* $\{(1.1), D\}$ *is LD if and only* $\xi_0(0) > 0$.

Proof. From (3.4) we have that

$$\xi_0(0) = \inf\{\mathcal{L}f : f \in D \cap U\}.$$

Then (3.2) holds for some $c > 0$ if and only if $\xi_0(0) > 0$. $\square$

We can now state our main existence result, its proof will be given in Section 4.

Theorem 3.1. *Let* $S(\lambda)$, $\lambda \in \mathbb{R}$, *be a self-adjoint realization of Eq.* (1.2) *defined by* (2.7) *with domain D and assume* $\{(1.1), D\}$ *is* LD. *Then for each* $n \in \mathbb{N}_0$, *the equation* $\xi_n(\lambda) = 0$ *always has exactly one positive root* $\lambda = \lambda_n$ *and one negative root* $\lambda = \lambda_{-n}$, *and they satisfy that* $|\lambda_{\pm n}| \geqslant \xi_n(0)$.

(i) *Suppose that for some* $n \in \mathbb{N}_0$, *0 is an eigenvalue of* $S(\lambda_n)$. *Then* λ_j, $j = 0, \ldots, n$, *are eigenvalues of the problem* $\{(1.1), D\}$ *and satisfy*

$$0 < \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n. \tag{3.5}$$

Moreover, these eigenvalues are the only points of the spectrum of $\{(1.1), D\}$ in $[0, \lambda_n]$.

(ii) *Suppose that for some* $m \in \mathbb{N}_0$, 0 *is an eigenvalue of* $S(\lambda_{-m})$. *Then* λ_{-j}, $j = 0, \cdots, m$, *are eigenvalues of the problem* $\{(1.1), D\}$ *and satisfy*

$$\lambda_{-m} \leqslant \cdots \leqslant \lambda_{-1} \leqslant \lambda_{-0} < 0. \tag{3.6}$$

Moreover, these eigenvalues are the only points of the spectrum of $\{(1.1), D\}$ *in* $[\lambda_{-m}, 0]$.

(iii) *Suppose that for some* $m, n \in \mathbb{N}_0$, 0 *is an eigenvalue of both* $S(\lambda_{-m})$ *and* $S(\lambda_n)$. *Then* $\{\lambda_j : \ j = -m, \cdots, -1, -0, 0, 1, \cdots, n\}$ *are eigenvalues of the problem* $\{(1.1), D\}$ *and satisfy*

$$\lambda_{-m} \leqslant \cdots \leqslant \lambda_{-1} \leqslant \lambda_{-0} < 0 < \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n. \tag{3.7}$$

Moreover, these eigenvalues are the only points of the spectrum of $\{(1.1), D\}$ *in* $[\lambda_{-m}, \lambda_n]$.

In (3.5)–(3.7) *only geometrically double eigenvalues appear twice. Strict inequalities hold throughout* (3.5)–(3.7) *whenever the* BC *is separated or non-real coupled, or at least one endpoint is LP*.

The following is an immediate consequence of Theorem 3.1.

Corollary 3.1. *Let* $S(\lambda)$ *be a self-adjoint realization of Eq.* (1.2) *with domain D and assume* $\{(1.1), D\}$ *is LD. For each* $n \in \mathbb{N}_0$, *let* λ_n *and* λ_{-n} *be the unique positive and negative roots of the equation* $\xi_n(\lambda) = 0$, *respectively. Suppose that for each* $n \in \mathbb{N}_0$, *there exists an* $m \in \mathbb{N}_0$ *such that* $m \geqslant n$ *and 0 is an eigenvalue of both* $S(\lambda_{\pm m})$. Then λ_j, $j \in \mathbb{Z}^*$, *are all eigenvalues of the problem* $\{(1.1), D\}$ *and satisfy*

$$\cdots \leqslant \lambda_{-2} \leqslant \lambda_{-1} \leqslant \lambda_{-0} < 0 < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots. \tag{3.8}$$

Moreover, these eigenvalues are the only points of the spectrum in (λ_-, λ_+), *where* $\lambda_\pm = \lim_{n\to\pm\infty} \lambda_n$. *In* (3.8) *only geometrically double eigenvalues appear twice. Strict inequalities hold throughout* (3.8) *whenever the BC is separated or non-real coupled, or at least one endpoint is LP*.

Remark 3.2. From Lemma 2.3 we see that

$$\xi_n(\lambda) \leqslant \inf \sigma_e(S(\lambda)) \quad \text{ for all } \lambda \in \mathbb{R}, \ n \in \mathbb{N}_0.$$

Thus if for some $n \in \mathbb{N}_0$ there exists $\lambda \in \mathbb{R}$ such that $\xi_n(\lambda) < \inf \sigma_e(S(\lambda))$, then $\xi_n(\lambda)$ is an eigenvalue of $S(\lambda)$. Therefore, the assumption of Theorem 3.1, (i), is satisfied if for some $n \in \mathbb{N}_0$, there exists $\lambda_n > 0$ such that $\xi_n(\lambda_n) = 0 < \inf \sigma_e(S(\lambda_n))$. Similarly, the assumption of Theorem 3.1, (ii), is satisfied if for some $m \in \mathbb{N}_0$, there exists $\lambda_{-m} < 0$ such that $\xi_m(\lambda_{-m}) = 0 < \inf \sigma_e(S(\lambda_{-m}))$. Furthermore, the assumption of Corollary 3.1 is satisfied if for each $n \in \mathbb{N}_0$, there exist $m \geqslant n$ and $\lambda_{-m} < 0$, $\lambda_m > 0$ such that $\xi_m(\lambda_{\pm m}) = 0 < \inf \sigma_e(S(\lambda_{\pm m}))$, respectively.

Theorem 3.2. *Let* $S(\lambda)$ *be a self-adjoint realization of Eq.* (1.2) *with domain D and assume* $\{(1.1), D\}$ *is* LD.

(i) *Suppose that for some* $n \in \mathbb{N}_0$, $\xi = \xi_n(\lambda)$, $\lambda \in \mathbb{R}$, *is an eigencurve of S. Then the conclusion of Theorem* 3.1, (iii) *holds with* $m = n$.

(ii) *Suppose that for each* $n \in \mathbb{N}_0$, $\xi = \xi_n(\lambda)$, $\lambda \in \mathbb{R}$, *is an eigencurve of S. Then the conclusion of Corollary* 3.1 *holds*.

Now, we state a result on the numbers of zeros of eigenfunctions for the LP case. A parallel result for the LC nonoscillatory case will be given in Section 5.

Corollary 3.2. *Let* $S(\lambda)$ *be a self-adjoint realization of Eq*. (1.2) *with domain D and assume* $\{(1.1), D\}$ *is LD. Suppose that at lease one of a and b is* LP, *and* $\lambda_n, n \in \mathbb{Z}^*$, *is the eigenvalue of the problem* $\{(1.1), D\}$ *with index n ensured by either Theorem* 3.1, *Corollary* 3.1, *or Theorem* 3.2. *Then any eigenfunction for* λ_n *has exactly* $|n|$ *zeros in the open interval J*.

Proof. Note that λ_n is the eigenvalue of $\{(1.1), D\}$ with index n if and only if 0 is the $|n|$th eigenvalue of $\{(1.2), D\}$ with $\lambda = \lambda_n$, and the two eigenvalues have the same eigenspace. Hence the conclusion follows from Theorem 14.10 in [27]. □

Remark 3.3. Whenever one or more eigencurves $\xi_n(\lambda)$ exist, Theorem 3.2 yields an algorithm for the numerical computation of the eigenvalues of general singular LD problems (with separated or coupled boundary conditions if needed): Use the Bailey et al. code SLEIGN2, see [1], to compute an eigencurve $\xi_n(\lambda)$, then use a root finder to locate its unique positive root λ_n and its unique negative root λ_{-n}. Note that, see Remark 3.2 above, if the eigencurve $\xi_n(\lambda)$ exists for some $n \in \mathbb{N}_0$, then all eigencurves $\xi_j(\lambda)$ exist for $j = 0, 1, ..., n$. The asymptotic behavior and the monotonicity properties of the eigencurves, see Lemmas 4.2 and 4.4 below, guarantee an efficient root finding scheme. In particular, when neither endpoint is LP, then all eigencurves $\xi_n(\lambda)$, $n \in \mathbb{N}_0$, exist. This case is studied in detail in Section 5.

Theorem 3.3. *Let S be a self-adjoint realization of Eq*. (1.3) *with domain D. Assume that* $\sigma(S)$ *is discrete and has a positive lower bound. Then, the* SLP $\{(1.1), D\}$ *is* LD, *its spectrum consists of only real eigenvalues, the eigenvalues are unbounded from below and above, and they can be ordered to satisfy*

$$\cdots \leqslant \lambda_{-2} \leqslant \lambda_{-1} \leqslant \lambda_{-0} < 0 < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$$

with only geometrically double eigenvalues appearing twice.

Proof. By definition, the SLP $\{(1.1), D\}$ is LD. From Lemma 3.2 we then deduce that its spectrum is real. Since $\sigma_e(S) = \emptyset$, Proposition 3.4 in [15] implies that the essential spectrum of the problem is also empty. So, the spectrum of the problem consists of real eigenvalues only. From Theorem 3.2 Part (ii) we know that the problem has an infinite number of positive and an infinite number of negative eigenvalues. Their multiplicities are either 1 or 2, and hence can be ordered to satisfy the inequalities in the theorem. Since the essential spectrum of the problem is empty, we have that $\lambda_{\pm n} \to \pm\infty$ as $n \to +\infty$, or equivalently in this case, the eigenvalues are unbounded from below and above. $\square$

The next results illustrate the applications of Theorem 3.3 to the existence of eigenvalues of LD problems with an LP endpoint.

For $\alpha \in [0, \pi)$, consider the SLP consisting of the differential equation

$$-y'' + qy = \lambda wy \quad \text{on } J = (0, \infty) \tag{3.9}$$

and the BC

$$\cos\alpha\, y(0) - \sin\alpha\, y'(0) = 0. \tag{3.10}$$

Theorem 3.4. *Assume*

(i) $q, w \in L_{\text{loc}}([0,\infty), \mathbb{R})$, *and w changes sign on J;*

(ii) *there exist $k, h > 0$ such that $q(t) \geqslant k$ a.e. on J and $\int_t^{t+h} q \to \infty$ as $t \to \infty$;*

(iii) *there exist $k_1, k_2 > 0$ such that $k_1 \leqslant |w(t)| \leqslant k_2$ a.e. on J.*

Then, for each $\alpha \in [0, \pi/2]$, the spectrum of the SLP (3.9), (3.10) *consists of only real eigenvalues. They are unbounded from below and above, and can be ordered to satisfy*

$$\cdots < \lambda_{-2} < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots.$$

Moreover, for every $n \in \mathbb{Z}^*$, each eigenfunction for λ_n has exactly $|n|$ zeros in the open interval J.

Proof. Consider the associated family of RD problems consisting of the equation

$$-y'' + (q - \lambda w)y = \xi|w|y \quad \text{on } J \tag{3.11}$$

and BC (3.10) with $\alpha \in [0, \pi)$. First note (3.9) is LP at infinity for each $\lambda \in \mathbb{R}$. This is well known for $|w| = 1$, see [10], and can be shown to hold for w bounded. In fact, the proof of Lemma 2.1 is still valid when the constant ξ in Eq. (1.3) is replaced by a bounded function. Let $S(\lambda, \alpha)$ denote the self-adjoint operator realization of SLP (3.11), (3.10) in $L^2(J, |w|)$ and denote its spectrum by $\sigma(S(\lambda, \alpha))$. Then $\sigma(S(\lambda, \alpha))$ is bounded below and discrete for each $\lambda \in \mathbb{R}$ and each $\alpha \in [0, \pi)$ by the extension of the well known Molchanov criterion to bounded weight functions, see Theorem 4 in Kwong and Zettl [18] (where f can be chosen as 1). Hence $\xi_n(\lambda)$ is an eigencurve for each $n \in \mathbb{N}_0$ and every $\alpha \in [0, \pi)$. Fix an $\alpha \in [0, \pi/2]$. Let g be a real eigenfunction of $\xi_0(0)$ normalized to satisfy $\int_0^\infty g^2|w| = 1$. By Theorem 1 of Everitt [13] we have that g' and $q^{1/2}g$ are in $L^2(J,1)$ and $g(t)g'(t) \to 0$ as $t \to \infty$. This and integration by parts yield that for $\alpha \in [0, \pi/2]$

$$\xi_0(0) = \int_0^\infty g[-g'' + qg] = g(0)g'(0) + \int_0^\infty g'^2 + \int_0^\infty qg^2 > g(0)g'(0) \geqslant 0.$$

Thus, $\sigma(S(0,\alpha))$ has a positive lower bound. By Theorem 3.3, the spectrum of the SLP (3.9), (3.10) consists of only real eigenvalues, which are unbounded from below and above, and can be ordered to satisfy the inequalities in Theorem 3.3. Since the eigenvalues all have multiplicity 1, they can be ordered to satisfy the inequalities in the theorem. The last conclusion of the theorem follows from Corollary 3.2. □

4. Lemmas and proofs

In this section we present several technical lemmas about the spectral-curves $\xi_n(\lambda)$, $n \in \mathbb{N}_0$, defined by (3.3), and then use these lemmas to prove Theorem 3.1.

Lemma 4.1. *Let $n \in \mathbb{N}_0$. For any $\lambda_*, \lambda_\# \in \mathbb{R}$ we have that*

$$|\xi_n(\lambda_*) - \xi_n(\lambda_\#)| \leqslant |\lambda_* - \lambda_\#|. \tag{4.1}$$

Hence $\xi_n(\lambda)$ is continuous on $\mathbb{R}$.

Proof. Note that for any $f \in U$

$$\mathcal{R}f = \int_a^b |f|^2 w \in [-1, 1].$$

Then (4.1) follows from (3.4). This implies that $\xi_n(\lambda)$ is continuous in $\mathbb{R}$. □

From Lemma 4.1 we obtain an interesting result on RD problems.

Corollary 4.1. *Let $S(\lambda)$ be a self-adjoint realization of Eq. (1.2) with domain D. Assume $\sigma(S(\lambda))$ is bounded below and discrete for some $\lambda \in \mathbb{R}$. Then $\sigma(S(\lambda))$ is bounded below and discrete for all $\lambda \in \mathbb{R}$.*

Proof. Let $\sigma(S(\lambda))$ be bounded below and discrete for $\lambda = \lambda^*$. Then $\sigma(S(\lambda))$ is bounded below for all $\lambda \in \mathbb{R}$ by Lemma 2.2. Note that $\sigma_e(S(\lambda^*)) = \emptyset$ means that

$\inf \sigma_e(S(\lambda^*)) = \infty$. Lemma 2.3 implies that for each $\lambda \in \mathbb{R}$,

$$\inf \sigma_e(S(\lambda)) = \lim_{n\to\infty} \xi_n(\lambda).$$

Thus $\lim_{n\to\infty} \xi_n(\lambda^*) = \infty$. From (4.1) we have that for each $\lambda \in \mathbb{R}$, $\lim_{n\to\infty} \xi_n(\lambda) = \infty$. Therefore, $\inf \sigma_e(S(\lambda)) = \infty$, i.e., $\sigma_e(S(\lambda)) = \emptyset$. $\square$

Lemma 4.2. *For $n \in \mathbb{N}_0$ we have that*

$$\lim_{\lambda\to\infty} \frac{\xi_n(\lambda)}{\lambda} = -1 \textit{ and } \lim_{\lambda\to-\infty} \frac{\xi_n(\lambda)}{\lambda} = 1. \tag{4.2}$$

Proof. Note that the definition of $\xi_0(0)$ implies that $\mathcal{L}f \geqslant \xi_0(0)$ for all $f \in D \cap U$. Thus, from (3.4) we obtain

$$\begin{aligned} \frac{\xi_n(\lambda)}{\lambda} &\geqslant \frac{\xi_0(0)}{\lambda} - \sup_{f\in D\cap U} \int_a^b |f|^2 w \geqslant \frac{\xi_0(0)}{\lambda} - \sup_{f\in D\cap U} \int_a^b |f|^2 |w| \\ &= \frac{\xi_0(0)}{\lambda} - 1 \longrightarrow -1 \text{ as } \lambda \longrightarrow \infty. \end{aligned}$$

Since w changes sign and $|w| > 0$ a.e. on J, there exist subsets J_1 and J_2 of J such that $w > 0$ on J_1, $w < 0$ on J_2, $m(J_j) > 0$, $j = 1, 2$, and $m(J) = m(J_1) + m(J_2)$, where m denotes Lebesgue measure. Let $\varepsilon > 0$. Since D is dense in H, we can choose an $(n+1)$-dimensional linear subspace F of D such that for all $f \in F \cap U$ we have

$$\int_{J_1} |f|^2 w > 1 - \varepsilon/2.$$

We observe that

$$1 = \int_J |f|^2 |w| = \int_{J_1} |f|^2 w - \int_{J_2} |f|^2 w.$$

Thus

$$\int_{J_2} |f|^2 w = \int_{J_1} |f|^2 w - 1 > -\varepsilon/2.$$

It follows that

$$\int_J |f|^2 w = \int_{J_1} |f|^2 w + \int_{J_2} |f|^2 w > 1 - \varepsilon.$$

Since F is finite dimensional, there exists an upper bound c for the set $\{\mathcal{L}f : f \in F \cap U\}$. Then from (3.4) we have

$$\frac{\xi_n(\lambda)}{\lambda} \leqslant \frac{c}{\lambda} - (1-\varepsilon) = \frac{c}{\lambda} - 1 + \varepsilon.$$

The first part of (4.2) follows from this. The proof of the second part is similar and hence omitted. □

Remark 4.1. We comment on the relationship between Lemma 4.2 and Theorem 2.2 in Binding and Volkmer [6]. For regular problems consisting of Eq. (1.2) and separated BC's with w bounded and $|w|$ replaced by 1, Theorem 2.2 in [6] gives an asymptotic result similar to (4.2) but with -1, $+1$ replaced by $-\sup w$ and $-\inf w$. Note that Lemma 4.2 does not require that w be bounded. Such an assumption is a relatively mild restriction in the regular case but a significant one for singular problems. Also, Lemma 4.2 applies to regular and singular equations, separated and coupled self-adjoint BC's. This illustrates one of the advantages in using $|w|$ instead of 1 on the right-hand side of Eq. (1.2). Moreover, (4.2) holds not only for eigencurves but also for spectral-curves.

Lemma 4.3. *Let $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$. Assume $\xi_0(0) > 0$ and $\xi_n(\lambda) < \xi_0(0)$. Let $\varepsilon \in [0, \xi_0(0) - \xi_n(\lambda))$. Then the set*

$$D_{\lambda,\varepsilon} := \{f \in D \cap U : \mathcal{L}f - \lambda\mathcal{R}f < \xi_0(0) - \varepsilon\} \tag{4.3}$$

is nonempty and for any $f \in D_{\lambda,\varepsilon}$ we have

$$\lambda\mathcal{R}f > \varepsilon. \tag{4.4}$$

Proof. Since $\xi_n(\lambda) < \xi_0(0) - \varepsilon$, $D_{\lambda,\varepsilon}$ is not empty by (3.4). Note that the definition of $\xi_0(0)$ implies that $\xi_0(0) \leqslant \mathcal{L}f$ for any $f \in D \cap U$. Thus for any $f \in D_{\lambda,\varepsilon}$

$$\mathcal{L}f - \lambda\mathcal{R}f < \mathcal{L}f - \varepsilon$$

and (4.4) follows. □

Lemma 4.4. *For $n \in \mathbb{N}_0$, $\xi_n(\lambda)$ is strictly decreasing in the region*

$$E_1 = \{(\lambda, \xi) : \lambda > 0,\ \xi < \xi_0(0)\}$$

and strictly increasing in the region

$$E_2 = \{(\lambda, \xi) : \lambda < 0,\ \xi < \xi_0(0)\}.$$

Proof. Let λ be such that $\xi_n(\lambda) < \xi_0(0)$, and let $\varepsilon \in [0, \xi_0(0) - \xi_n(\lambda))$. From (3.4)

$$\xi_n(\lambda) = \inf \left\{ \sup\{\mathcal{L}f - \lambda \mathcal{R}f : f \in F \cap D_{\lambda,\varepsilon}\} : F \in G_{n+1}(D) \right\},$$

where $D_{\lambda,\varepsilon}$ is defined by (4.3). Let λ_1 and λ_2 be such that $0 < \lambda_1 < \lambda_2$ and $(\lambda_1, \xi_n(\lambda_1))$ and $(\lambda_2, \xi_n(\lambda_2))$ are in E_1. Then from (4.3) and (4.4) we see that for $\varepsilon \in [0, \xi_0(0) - \xi_n(\lambda_1))$

$$D_{\lambda_1,\varepsilon} \subset D_{\lambda_2,\varepsilon} \subset D \cap U.$$

Thus, for $i = 1$ and 2,

$$\xi_n(\lambda_i) = \inf \left\{ \sup\{\mathcal{L}f - \lambda_i \mathcal{R}f : \ f \in F \cap D_{\lambda_2,\varepsilon}\} : \ \ F \in G_{n+1}(D) \right\}.$$

For any $f \in D_{\lambda_2,\varepsilon}$, by Lemma 4.3 we have that

$$\mathcal{R}f = \int_a^b |f|^2 w > \frac{\varepsilon}{\lambda_2},$$

and hence

$$(\mathcal{L}f - \lambda_2 \mathcal{R}f) - (\mathcal{L}f - \lambda_1 \mathcal{R}f) = (\lambda_1 - \lambda_2)\mathcal{R}f < \frac{\lambda_1 - \lambda_2}{\lambda_2}\varepsilon.$$

This implies that

$$\xi_n(\lambda_2) - \xi_n(\lambda_1) \leqslant \frac{\lambda_1 - \lambda_2}{\lambda_2}\varepsilon < 0,$$

and thus $\xi_n(\lambda_2) < \xi_n(\lambda_1)$.

The proof of the second part is similar and hence omitted. □

Proof of Theorem 3.1. Let $n \in \mathbb{N}_0$. From Lemma 3.3 we have that $\xi_n \geqslant \xi_0(0) > 0$. By Lemmas 4.1, 4.2 and 4.4, the nth spectral-curve $\xi = \xi_n(\lambda)$ is continuous on $\mathbb{R}$, strictly decreasing in E_1, strictly increasing in E_2, and $\lim_{\lambda \to \pm\infty} \xi_n(\lambda) = -\infty$. Therefore, the equation $\xi_n(\lambda) = 0$ has exactly one positive root $\lambda = \lambda_n$ and one negative root $\lambda = \lambda_{-n}$. Note that $\xi_n(\lambda) > 0$ for $\lambda \in (\lambda_{-n}, \lambda_n)$. Letting $\lambda_* = \lambda_{\pm n}$ and $\lambda_\# = 0$ in (4.1), we obtain the inequality $|\lambda_{\pm n}| \geqslant \xi_n(0)$.

(i) Since $0 = \xi_n(\lambda_n)$ is an eigenvalue of $S(\lambda_n)$ by assumption, we know that λ_n is an eigenvalue of $\{(1.1), D\}$.

If $\lambda_j = \lambda_n$ for some $j \in \{0, 1, \cdots, n-1\}$, then $\lambda_j = \lambda_n$ is an eigenvalue of $\{(1.1), D\}$; if $\lambda_j < \lambda_n$ for some $j \in \{0, 1, \cdots, n-1\}$, then

$$\xi_j(\lambda_j) = 0 < \xi_n(\lambda_j) \leqslant \inf \sigma_e(S(\lambda_j)),$$

which, together with Lemma 2.3, implies that $0 = \xi_j(\lambda_j)$ is an eigenvalue of $S(\lambda_j)$, and hence λ_j is also an eigenvalue of $\{(1.1), D\}$. Obviously, (3.5) holds.

Let $\lambda_* \in (0, \lambda_n) \setminus \{\lambda_0, \lambda_1, \cdots, \lambda_{n-1}\}$. Then $0 \neq \xi_i(\lambda_*)$ for $i = 0, 1, ..., n-1$. Since $0 < \xi_n(\lambda_*) \leqslant \inf \sigma_e(S(\lambda_*))$, Lemma 2.3 implies that $0 \notin \sigma(S(\lambda_*))$. Thus, by Lemma 3.1, λ_* is not in the spectrum of $\{(1.1), D\}$. Hence, from $[0, \lambda_n]$, λ_0, λ_1, ..., λ_n are the only points in the spectrum of $\{(1.1), D\}$.

Since $\lambda_j = \lambda_{j+1}$ for some $j \in \{0, 1, \ldots, n-1\}$ if and only if $0 = \xi_j(\lambda_j) = \xi_{j+1}(\lambda_{j+1})$ is an eigenvalue of $S(\lambda_j) = S(\lambda_{j+1})$ with multiplicity 2, it happens if and only if $\lambda_j = \lambda_{j+1}$ is an eigenvalue of $\{(1.1), D\}$ with multiplicity 2. Thus, in (3.5) only double eigenvalues appear twice.

(ii) The proof is similar to (i) and hence omitted.

(iii) This is an immediate consequence of Parts (i) and (ii). $\square$

Remark 4.2. Assume $\{(1.1), D\}$ is LD. Recall that for each $n \in \mathbb{N}_0$, λ_n and λ_{-n} are the positive and negative roots of the equation $\xi_n(\lambda) = 0$, respectively. Then the sequence $\{\lambda_n\}_{n=0}^{+\infty}$ is increasing, and $\{\lambda_{-n}\}_{n=0}^{+\infty}$ is decreasing. Set

$$\delta_\pm = \lim_{n \to \pm\infty} \lambda_n.$$

We have that

$$-\infty \leqslant \delta_- < 0 < \delta_+ \leqslant +\infty.$$

Theorem 3.1 implies that (δ_-, δ_+) does not intersect the essential spectrum of the problem, and all the eigenvalues in (δ_-, δ_+) can be obtained from spectral-curves. If δ_- is finite and is an eigenvalue, and the problem only has finitely many eigenvalues in $(\delta_-, 0)$, then δ_- can also be obtained from spectral-curves. There is a similar statement for δ_+.

5. LC non-oscillatory problems

In this section we specialize to the case where Eq. (1.1) is LC non-oscillatory. First, we clarify the concept of LC non-oscillation for LD problems, and show that every singular LD problem associated with an LC non-oscillatory equation can be transformed into a regular LD problem with the same spectrum. This extends the results in Niessen and Zettl [22] for singular LC non-oscillatory problems with positive w. Based on it and the recent work of Kong et al. [17] on regular LD problems, we present further results on the existence of the eigenvalues and the numbers of zeros of eigenfunctions.

Definition 5.1. For a fixed $\lambda \in \mathbb{R}$, the endpoint a for Eq. (1.1) is said to be *oscillatory*, or simply a is O, if every real-valued solution of (1.1) has an infinite number of zeros in (a, c) for any $c \in J$, and it is *non-oscillatory*, or simply a is NO, otherwise. LCO means LC and O, LCNO means LC and NO. Similar definitions are made for the endpoint b. Eq. (1.1) is LCNO if both a and b are LCNO.

It is well-known that for the case when w does not change sign, Eq. (1.1) is LCNO for some $\lambda \in \mathbb{R}$ if and only if it is LCNO for all $\lambda \in \mathbb{R}$, and Eq. (1.1) can be regularized using a so called "regularizing function". We now extend these results to the case for indefinite w.

Lemma 5.1. *Let Eq.* (1.1) *be LCNO for some $\lambda = \lambda_0 \in \mathbb{R}$. Then there exist real-valued functions u and v in the maximal domain $D_{\max}$ satisfying the following conditions*:

(a) $v > 0$ *on* $J = (a, b)$;

(b) *for some $\xi_1 \in \mathbb{R}$ and $\{\lambda, \xi\} = \{\lambda_0, \xi_1\}$, u is a principal solution of Eq.* (1.2) *at a and v is a non-principal solution of Eq.* (1.2) *at a*;

(c) *for some $\xi_2 \in \mathbb{R}$ and $\{\lambda, \xi\} = \{\lambda_0, \xi_2\}$, u is a principal solution of Eq.* (1.2) *at b and v is a non-principal solution of Eq.* (1.2) *at b*;

(d) $[u, v](a) = [u, v](b) = -1$.

Proof. We observe that Eq. (1.1) is LCNO for $\lambda = \lambda_0$ means that Eq. (1.2) is LCNO for $\{\lambda, \xi\} = \{\lambda_0, 0\}$. Note from Lemma 2.1 that $D_{\max}$ is the maximum domain of Eq. (1.2) with $\lambda = \lambda_0$. Then the conclusions follow from those for RD problems. See also [22, p. 564–566]. $\square$

The function v in Lemma 5.1 is called a *regularizing function* for Eq. (1.1), and as in Remark 2.3, $\{u, v\}$ is called a *BC basis* for Eq. (1.1).

Corollary 5.1. *Let $\{u, v\}$ be a BC basis for Eq.* (1.1) *defined as in Lemma* 5.1. *Then for any function $y \in D_{\max}$*

$$\frac{y}{v}(a) = [y, u](a), \qquad \frac{y}{v}(b) = [y, u](b).$$

Proof. For any $y \in D_{\max}$

$$\begin{aligned}
[y, u] &= y(pu') - u(py') \\
&= \frac{y}{v}[v(pu') - u(pv')] - \frac{u}{v}[v(py') - y(pv')] \\
&= \frac{y}{v}[v, u] - \frac{u}{v}[v, y].
\end{aligned}$$

Let $t \to a^+$, and noting that $\lim_{t\to a^+}(u/v)(t) = 0$ and $[v, u](a) = 1$, we obtain the equality for a. Similarly, we obtain the equality for b. □

Lemma 5.2. *Assume Eq.* (1.1) *is* LCNO *for some* $\lambda = \lambda_0 \in \mathbb{R}$ *and let* v *be a regularizing function for Eq.* (1.1). *Consider the equation*

$$-(Pz')' + Qz = \lambda W z \quad on\ J, \tag{5.1}$$

where

$$P = pv^2,\ Q = v[-(pv')' + qv],\ W = wv^2. \tag{5.2}$$

Then $P > 0$, $W \neq 0$ *a.e. on* J, $1/P,\ Q,\ W \in L^1(J, \mathbb{R})$. *Hence Eq.* (5.1) *is regular on* J. *Furthermore,* y *is a solution of Eq.* (1.1) *if and only if* $z = y/v$ *is a solution of Eq.* (5.1), *and in this case,* $Pz' = -[y, v]$.

Proof. This is a minor modification of Theorem 4.3 in [16] for the case with indefinite w and has a similar proof. □

Lemma 5.3. *Assume Eq.* (1.1) *is* LCNO *for some* $\lambda = \lambda_0 \in \mathbb{R}$. *Then Eq.* (1.1) *is* LCNO *for all* $\lambda \in \mathbb{R}$.

Proof. Let v be a regularizing function for Eq. (1.1). For any $\lambda \in \mathbb{R}$, let y be any nontrivial real-valued solution of Eq. (1.1). Lemma 5.2 shows that $z = y/v$ is a solution of the regular equation (5.1). Thus, the limits $\lim_{t\to a+} z(t)$ and $\lim_{t\to b-} z(t)$ exist and z is not oscillatory at either a or b. Therefore, $y = zv \in H$ and is not oscillatory at a and at b since $v \in H$ and $v > 0$ on J. Hence Eq. (1.1) is LCNO for all $\lambda \in \mathbb{R}$. □

Remark 5.1. By Lemma 5.3, LCNO is a property of Eq. (1.1) independent of λ even when w is indefinite. So we can simply say that Eq. (1.1) is LCNO if Eq. (1.1) is LCNO for some (and hence for all) $\lambda \in \mathbb{R}$.

Remark 5.2. Lemma 5.3 implies that if Eq. (1.1) is LCO for all $\lambda \in \mathbb{R}$, so is Eq. (1.3) for all $\xi \in \mathbb{R}$. In this case, the spectrum of every self-adjoint realization of Eq. (1.3) is unbounded above and below, see Niessen and Zettl [22]. Therefore, there are no LD problems associated with Eq. (1.1) in the LCO case.

For the rest of this paper, we assume Eq. (1.1) *is* LCNO *and* u, v *are chosen to satisfy the assumptions of Lemma* 5.1.

We observe from Proposition 2.1, (4), that all BC's determining self-adjoint domains for Eq. (1.1) are of the form

$$AY(a) + BY(b) = 0, \;\; Y = \begin{pmatrix} [y, u] \\ [y, v] \end{pmatrix}, \tag{5.3}$$

where A, B satisfy (2.8). In particular, the separated and coupled BC's have the canonical representations (2.9) and (2.10), respectively.

Lemma 5.4. *The singular* SLP (1.1), (5.3) *has exactly the same set of eigenvalues as the regular* SLP *consisting of Eq.* (5.1) *and the BC*

$$AZ(a) + BZ(b) = 0, \;\; Z = \begin{pmatrix} z \\ Pz' \end{pmatrix}. \tag{5.4}$$

and the eigenfunctions are related by $y_n = vz_n, n \in \mathbb{Z}^*$. *In particular, the eigenfunctions of the singular problem* (1.1), (5.3) *have exactly the same zeros as the eigenfunctions of the corresponding regular problem in* J.

Proof. From Lemmas 5.1 and 5.2, the transformation $z = y/v$ changes the problem (1.1), (5.3) into the problem (5.1), (5.4) and vice versa. Thus, the two problems have the same set of eigenvalues. Note that for some eigenvalue λ, y is an eigenfunction of (1.1), (5.3) if and only if $z = y/v$ is an eigenfunction of (5.1), (5.4). Hence y and z have the same zeros in J. $\square$

Next, we investigate the relationship between the left-definiteness of singular and regular problems.

Theorem 5.1. *The singular* SLP (1.1), (5.3) *is* LD *if and only if the corresponding regular* SLP (5.1), (5.4) *is LD*.

Proof. Let P, Q, W be given in (5.2). Define U and $\tilde{U}$ to be the unit spheres in $L^2(J, |w|)$ and $L^2(J, |W|)$, respectively; and D and $\tilde{D}$ the self-adjoint domains in $L^2(J, |w|)$ and $L^2(J, |W|)$ characterized by (5.3) and (5.4), respectively. Let the functional $\mathcal{L}$ on D be defined as in (3.1), and the functional $\tilde{\mathcal{L}}$ on $\tilde{D}$ be defined by

$$\tilde{\mathcal{L}}g = \int_a^b [-(Pg')'\bar{g} + Q|g|^2].$$

It is easy to see that $f \in D \cap U$ if and only if $g = f/v \in \tilde{D} \cap \tilde{U}$, and for $f \in D \cap U$ and $g = f/v$ we have

$$-(Pg')'\bar{g} + Q|g|^2 = -(pf')'\bar{f} + q|f|^2.$$

By the definition of left-definiteness of regular SLP's (see [17]), SLP (5.1), (5.4) is LD if and only if there exists $c > 0$ such that $\tilde{\mathcal{L}}g \geqslant c > 0$ for all $g \in \tilde{D} \cap \tilde{U}$, hence if and only if $\mathcal{L}f \geqslant c > 0$ for all $f \in D \cap U$. This means that the left-definiteness of the regular SLP (5.1), (5.4) is equivalent to that of the singular SLP (1.1), (5.3). □

Corollary 5.2. *Assume there exists a regularizing function v given in Lemma* 5.1 *such that Q defined in* (5.2) *satisfies* $Q(t) \geqslant 0$ *and* $Q(t) \not\equiv 0$ *a.e. on J. Then*

(i) *the separated* SLP (1.1), (2.9) *is LD if* $\pi/2 \leqslant \alpha \leqslant \pi$ *and* $0 \leqslant \beta \leqslant \pi/2$;

(ii) *the coupled* SLP (1.1), (2.10) *is LD if* $K = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}$ *for some real number* $c \neq 0$.

Assume there exists a regularizing function v given in Lemma 5.1 *such that Q defined in* (5.2) *satisfies* $Q = 0$ *a.e. on J. Then*

(i) *the separated* SLP (1.1), (2.9) *is LD if* $\pi/2 \leqslant \alpha \leqslant \pi$, $0 \leqslant \beta \leqslant \pi/2$, *and* $(\alpha, \beta) \notin \{(0, \pi/2), (0, \pi), (\pi/2, \pi/2), (\pi/2, \pi)\}$;

(ii) *the coupled* SLP (1.1), (2.10) *is LD if* $K = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}$ *for some real number* $c \neq 0$, and $ce^{i\gamma} \neq 1$.

Proof. The first part follows directly from Theorem 5.1 and the results in Corollary 2.1, [17], on regular LD SLP's. The second part can be justified by a slight modification of the proof of that corollary. □

Remark 5.3. Since the regularizing function v is not explicitly given, the hypotheses on Q in Corollary 5.1 may be difficult to verify. However, we observe the following: if there exists $r > 0$ such that Eq. (1.2) with $\{\lambda, \xi\} = \{0, r\}$ has a positive solution v which is non-principal at both endpoints a and b, then v can be chosen as a regularizing function. In this case, $Q = vMv = r|w|v^2 \geqslant 0$ and $Q \not\equiv 0$ on J. If Eq. (1.2) with $\{\lambda, \xi\} = \{0, 0\}$ has a positive solution v which is non-principal at both endpoints a and b, then v can be chosen as a regularizing function. In this case, $Q = vMv = 0$.

Example 1. Consider the equation

$$-y'' + \frac{\nu^2 - 1/4}{t^2} y = \lambda w y \quad \text{on } J = (0, 1), \tag{5.5}$$

where $0 < \nu < 1, \nu \neq 1/2$, $w \in L(J; \mathbb{R})$, w changes sign and $|w| = 1$ a.e. on J.

It is easy to see that Eq. (5.5) is LCNO at 0 and regular at 1. The right-definite equation associated with Eq. (5.5) is the Bessel equation

$$-y'' + \frac{\nu^2 - 1/4}{t^2} y = \xi y \quad \text{on } J. \tag{5.6}$$

For $\xi = 0$,

$$u_1(t) = -\frac{1}{2\nu} t^{1/2+\nu} \quad \text{and} \quad v(t) = t^{1/2-\nu}$$

are principal and non-principal solutions, respectively, of Eq. (5.6) at 0; and $u_2(t) := u_1(t) + \frac{1}{2\nu} v(t)$ and $v(t)$ are principal and non-principal solutions, respectively, of Eq. (5.6) at 1. Let $u \in D_{\max}$ be defined by u_1 on $(0, c]$, u_2 on $[d, 1)$, and by the Naimark Lemma on (c, d), where $0 < c < d < 1$. Then u and v satisfy the assumptions of Lemma 5.1. Using $\{u, v\}$ as a BC basis, we obtain the separated BC

$$\begin{aligned} \cos\alpha\, [y, u](0) - \sin\alpha\, [y, v](0) &= 0 \\ \cos\beta\, [y, u](1) - \sin\beta\, [y, v](1) &= 0. \end{aligned} \tag{5.7}$$

We claim that SLP (5.5), (5.7) is LD provided $\pi/2 \leqslant \alpha \leqslant \pi$, $0 \leqslant \beta \leqslant \pi/2$, and $(\alpha, \beta) \notin \{(0, \pi/2), (0, \pi), (\pi/2, \pi/2), (\pi/2, \pi)\}$. In fact, Let $Q = vMv$. Then $Q \equiv 0$ by Remark 5.1, and hence the conclusion follows from Corollary 5.1.

Similarly, for the coupled BC

$$Y(1) = e^{i\gamma} K Y(0), \quad Y = \begin{pmatrix} [y, u] \\ [y, v] \end{pmatrix}, \tag{5.8}$$

where $K = \begin{pmatrix} c & 0 \\ 0 & 1/c \end{pmatrix}$ for some real number $c \neq 0$, $\gamma \in [0, \pi)$, SLP (5.5), (5.8) is LD if $ce^{i\gamma} \neq 1$.

The next result gives more information on the existence of eigenvalues and the numbers of zeros of eigenfunctions for the LCNO case.

Theorem 5.2. *Assume* SLP (1.1), (5.3) *is* LD *and* LCNO. *Then its spectrum consists of only real eigenvalues, the eigenvalues are unbounded from below and above, and can be ordered to satisfy the inequalities*

$$\cdots \leqslant \lambda_{-2} \leqslant \lambda_{-1} \leqslant \lambda_{-0} < 0 < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \tag{5.9}$$

with only geometrically double eigenvalues appearing twice. Furthermore, we have the following:

(i) *If the* BC (5.3) *becomes the separated* BC (2.9), *then all inequalities in* (5.9) *are strict. For* $n \in \mathbb{Z}^*$, *any eigenfunction* u_n of λ_n *has exactly* $|n|$ *zeros in the open interval* J.

(ii) *If the* BC (5.3) *becomes the real coupled self-adjoint* BC, *i.e.*, (2.10) *with* $\gamma = 0$, *then the equalities in* (5.9) *may occur. For* $n \in \mathbb{Z}^*$, *Let* u_n *be a real-valued eigenfunction of* λ_n, *then the number of zeros of* u_n *in the interval* $[a, b)$ *is 0 or 1 if* $n = \pm 0$ *or* ± 1 *and is* $|n| - 1$ *or* $|n|$ *or* $|n| + 1$ *when* $|n| > 1$.

(iii) *If the* BC (5.3) *becomes the non-real coupled self-adjoint* BC, *i.e.*, (2.10) *with* $\gamma \neq 0$, *then all inequalities in* (5.9) *are strict. For* $n \in \mathbb{Z}^*$, *let* u_n *be an eigenfunction of* λ_n, *then the number of zeros of* $\mathrm{Re}\,u_n$ *and of* $\mathrm{Im}\,u_n$ *in the interval* $[a, b)$ *is 0 or 1 if* $n = \pm 0$ *or* ± 1 *and is* $|n| - 1$ *or* $|n|$ *or* $|n| + 1$ *when* $|n| > 1$, *and* u_n *itself has no zero in* $[a, b]$.

Proof. From Theorem 5.1 and Lemma 5.4, we see that SLP (1.1), (5.3) is LD implies that SLP (5.1), SLP (5.4) is LD, the two problems have exactly the same eigenvalues, and the corresponding eigenfunctions have the same zeros in J. Then the conclusions follow from Theorems 3.1 and 3.2 in [17]. □

Example 2. Consider the equation

$$-((1-t^2)y')' + y = \lambda w y \quad \text{on } J = (-1, 1), \tag{5.10}$$

where $|w| = 1$ and w changes sign on J. The RD equation associated with Eq. (5.10) is the classical Legendre equation

$$-((1-t^2)y')' + y = \lambda y \quad \text{on } J = (-1, 1). \tag{5.11}$$

Note that both endpoints are LCNO and $u \equiv 1$ is a principal solution of Eq. (5.11) for $\lambda = 1$ at both endpoints of J. Then

$$[y, 1](-1) = 0 = [y, 1](1) \tag{5.12}$$

defines a self-adjoint separated BC and SLP (5.11), (5.12) is the classical self-adjoint Legendre SLP. Its eigenvalues are known to be

$$\lambda_n = 1 + n(n+1), \ n \in \mathbb{N}_0,$$

and the eigenfunctions are the classical Legendre polynomials. Since $\lambda_0 = 1 > 0$, the problem consisting of Eq. (5.10) and the same BC (5.12) is LD. Hence by Theorem 5.2, eigenvalues of the problem (5.10), (5.12) are all real, unbounded from below and above, and satisfy inequalities (5.9) with strict inequality holding everywhere; and if

u_n is an eigenfunction of λ_n for some $n \in \mathbb{Z}^*$, then u_n has exactly $|n|$ zeros in the open interval J.

6. Further eigenvalue properties in LCNO case

Finally, we present results on eigenvalue inequalities, asymptotic behavior and ranges of eigenvalues for singular LD problems with LCNO endpoints, which are analogues of Theorems 3.3–3.5 in [17] for regular LD problems. Such general eigenvalue inequalities have recently been established by Eastham, Kong et al. in [11] for regular RD problems; by Kong et al. [16] for singular RD problems; and by Kong et al., in [17] for regular LD problems; respectively. The proofs follow from the regular case, Lemma 5.4 and Theorem 5.1. We omit the details.

In this section, we assume a and b are LCNO endpoints. For $K = (k_{ij}) \in SL(2, \mathbb{R})$, we denote by $\{\mu_n : n \in \mathbb{Z}^*\}$ and $\{\nu_n : n \in \mathbb{Z}^*\}$ the eigenvalues of Eq. (1.1) and the separated BC's

$$[y, u](a) = 0, \;\; k_{22}[y, u](b) - k_{12}[y, v](b) = 0 \tag{6.1}$$

and

$$[y, v](a) = 0, \;\; k_{21}[y, u](b) - k_{11}[y, v](b) = 0, \tag{6.2}$$

respectively, and by $\{\lambda_n(e^{i\gamma}K) : n \in \mathbb{Z}^*\}$ the eigenvalues of Eq. (1.1) and the coupled BC

$$Y(b) = e^{i\gamma}K\,Y(a), \;\; Y = \begin{pmatrix} [y, u] \\ [y, v] \end{pmatrix}. \tag{6.3}$$

Theorem 6.1. *Let $K = (k_{ij}) \in SL(2, \mathbb{R})$.*

(a) *Assume that $k_{11} > 0$, $k_{12} \leqslant 0$, and* SLP (1.1), (6.2) *is* LD. *Then, both problem* (1.1), (6.1) *and problem* (1.1), (6.3) *with any $\gamma \in (-\pi, \pi]$ are* LD. *Furthermore, $\lambda_{\pm 0}(K)$ are geometrically simple, and for each $\gamma \in (-\pi, \pi)$, $\gamma \neq 0$, we have*

$$\begin{aligned} \nu_0 \;&\leqslant \lambda_0(K) < \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leqslant \{\mu_0, \nu_1\} \\ &\leqslant \lambda_1(-K) < \lambda_1(e^{i\gamma}K) < \lambda_1(K) \leqslant \{\mu_1, \nu_2\} \\ &\leqslant \lambda_2(K) < \lambda_2(e^{i\gamma}K) < \lambda_2(-K) \leqslant \{\mu_2, \nu_3\} \\ &\leqslant \lambda_3(-K) < \lambda_3(e^{i\gamma}K) < \lambda_3(K) \leqslant \{\mu_3, \nu_4\} \leqslant \cdots \end{aligned} \tag{6.4}$$

and another set of inequalities obtained by replacing λ_n, μ_n, ν_n, $<$ and $\leqslant$ in (6.4) *by λ_{-n}, μ_{-n}, ν_{-n}, $>$ and $\geqslant$, respectively.*

(b) *Assume that $k_{11} \leqslant 0$, $k_{12} < 0$, and SLP* (1.1), (6.3) *with $\gamma = 0$ is LD. Then problem* (1.1), (6.3) *with any other $\gamma \in (-\pi, \pi]$, problem* (1.1), (6.1), *and problem*

(1.1), (6.2) *are all LD. Furthermore,* $\lambda_{\pm 0}(K)$ *are geometrically simple, and for each* $\gamma \in (-\pi, \pi)$, $\gamma \neq 0$, *we have*

$$\begin{aligned} &\lambda_0(K) < \lambda_0(e^{i\gamma}K) < \lambda_0(-K) \leqslant \{\mu_0, \nu_0\} \leqslant \\ &\lambda_1(-K) < \lambda_1(e^{i\gamma}K) < \lambda_1(K) \leqslant \{\mu_1, \nu_1\} \leqslant \\ &\lambda_2(K) < \lambda_2(e^{i\gamma}K) < \lambda_2(-K) \leqslant \{\mu_2, \nu_2\} \leqslant \\ &\lambda_3(-K) < \lambda_3(e^{i\gamma}K) < \lambda_3(K) \leqslant \{\mu_3, \nu_3\} \leqslant \cdots \end{aligned} \tag{6.5}$$

and another set of inequalities obtained by replacing λ_n, μ_n, ν_n, $<$ *and* $\leqslant$ *in* (6.5) *by* $\lambda_{-n}, \mu_{-n}, \nu_{-n}$, $>$ *and* $\geqslant$, *respectively.*

(c) *If neither case* (a) *nor case* (b) *applies to K, then either case (a) or case (b) applies to* $-K$.

Theorem 6.2. *Assume* SLP (1.1), (5.3) *is* LD, *and denote its eigenvalues by* $\{\lambda_n : n \in \mathbb{Z}^*\}$. *Then*

$$\lambda_{\pm n} \sim \pm \frac{n^2\pi^2}{\left[\int_a^b \sqrt{\frac{w_\pm(t)}{p(t)}}\, dt\right]^2}, \quad n \to \infty,$$

where w_+ *and* w_- *denote the positive and negative parts of* w, *respectively.*

The eigenvalues for the BC

$$[y, u](a) = 0 = [y, u](b) \tag{6.6}$$

play a special role in determining the upper and lower bounds of the eigenvalues of SLP (1.1), (5.3) for all self-adjoint BC's. In the RD case (6.6) is called the Friedrichs BC since it determines the so called Friedrichs extension. In the LD case there is no Friedrichs extension since the spectrum is not bounded below. Nevertheless, in analogy with the RD case we will denote the eigenvalues for (6.6) by $\{\lambda_n^F : n \in \mathbb{Z}^*\}$.

Theorem 6.3. *Assume* SLP (1.1), (5.3) *is* LD, *and denote its eigenvalues by* $\{\lambda_n : n \in \mathbb{Z}^*\}$. *Then* SLP (1.1), (6.6) *is* LD, *and*

$$\lambda_n \in (0, \lambda_n^F], \quad \lambda_{-n} \in [\lambda_{-n}^F, 0) \quad \text{for } n = 0, 1,$$

$$\lambda_n \in (\lambda_{n-2}^F, \lambda_n^F], \quad \lambda_{-n} \in [\lambda_{-n}^F, \lambda_{-n+2}^F) \quad \text{for } n = 2, 3, 4, \ldots\ .$$

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